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Equation for Additive Drag Coefficient at Static Conditions

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Introduction

CALCULATION of additive drag at static conditions is usually not a problem simply because multiplying the drag coefficient by the dynamic pressure (which is zero at static conditions) is always zero. However, in some computer codes, if the additive drag coefficient is determined by dividing the additive drag force by the dynamic pressure, a zero divide error can result at static conditions. This problem is eliminated if the drag coefficient at static conditions is determined independently of the dynamic pressure. This technical Note describes how such an expression was derived from an existing additive drag coefficient equation.

Procedure

To use freestream velocity in the determination of airbreathing propulsion thrust, the system boundary is extended forward of the inlet cowl so that freestream velocity air crosses the system boundary. This requires that the drag component attributable to the airstream ahead of the inlet is included in determining the overall engine thrust. The nomenclature as used in this Note and in Ref. 1 is shown in Fig. 1.

The additive drag coefficient is calculated in Ref. 1 from

$$Cd_{\text{pre}} = \left[(P_c - P_\infty)A_c + \rho_c V_c^2 A_c - \rho_\infty V_\infty^2 A_\infty\right] \big/ q_\infty A_c \qquad (1)$$

where P_c is static pressure at the cowl lip, P_∞ is freestream static pressure, A_c is the cowl lip area, ρ_c is the air density at the cowl lip, V_c is the air velocity at the cowl lip, ρ_∞ is the freestream air density, V_∞ is the freestream air velocity, A_∞ is the capture area of freestream flow, and q_∞ is the freestream dynamic pressure. However, because $q_\infty = 0$ when $M_\infty = 0$, this expression becomes indeterminate at static conditions. Yet the plot of Cd_{pre} as a function of A_∞/A_c and M_∞ (see page 196 in Ref. 1) shows finite values of Cd_{pre} values for $M_\infty = 0$. Therefore, all of the variables on the right side of Eq. (1) should be definable in terms of A_∞/A_c and M_∞. However, as will be shown, the desired expression of Cd_{pre} as a function of A_∞/A_c and M_∞ only is not possible. Ultimately, application of L'Hopital's rule will be required to obtain the desired expression for Cd_{pre} at static conditions.

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The first step is to rewrite Eq. (1) as the sum of three terms

$$Cd_{\text{pre}} = \frac{(P_c - P_\infty)A_c}{q_\infty A_c} + \frac{\rho_c V_c^2 A_c}{q_\infty A_c} - \frac{\rho_\infty V_\infty^2 A_\infty}{q_\infty A_c} \qquad (2)$$

which is simplified to

$$Cd_{\text{pre}} = \frac{P_c}{q_\infty} - \frac{P_\infty}{q_\infty} + \frac{\rho_c V_c^2}{q_\infty} - \frac{\rho_\infty V_\infty^2 A_\infty}{q_\infty A_c} \qquad (3)$$

Note that A_∞/A_c now appears explicitly in the third term. Also

$$q_\infty = \tfrac{1}{2}\rho_\infty V_\infty^2, \qquad \rho_c V_c^2 = \gamma M_c^2 P_c \qquad (4)$$

Therefore, substituting Eq. (4) into Eq. (3) gives

$$Cd_{\text{pre}} = \frac{P_c}{\frac{1}{2}P_\infty \gamma M_\infty^2} - \frac{P_\infty}{\frac{1}{2}P_\infty \gamma M_\infty^2} + \frac{\gamma P_c M_c^2}{\frac{1}{2}P_\infty \gamma M_\infty^2} - \frac{\rho_\infty V_\infty^2 A_\infty}{\frac{1}{2}P_\infty \gamma M_\infty^2 A_c} \qquad (5)$$

which becomes

$$Cd_{\text{pre}} = \frac{2P_c}{P_\infty \gamma M_\infty^2} - \frac{2}{\gamma M_\infty^2} + \frac{2P_c M_c{}^2}{P_\infty M_\infty^2} - 2\left(\frac{A_\infty}{A_c}\right) \qquad (6)$$

or

$$Cd_{\text{pre}} = \frac{2}{\gamma M_\infty^2}\left(\frac{P_c}{P_\infty} - 1\right) + \frac{2P_c M_c^2}{P_\infty M_\infty^2} - 2\left(\frac{A_\infty}{A_c}\right) \qquad (7)$$

Next, note that

$$P_c/P_\infty = (P_c/P_T)/(P_\infty/P_T) \qquad (8)$$

where P_T is the stagnation pressure for isentropic flow from freestream conditions to the cowl entrance. Then, because the ratio of static to total pressure for isentropic flow can be written as

$$P/P_T = \{1 + [(\gamma - 1)/2]M^2\}^{\gamma/(1-\gamma)} \qquad (9)$$

therefore,

$$\frac{P_c}{P_\infty} = \frac{\left\{1 + [(\gamma - 1)/2]M_c^2\right\}^{\gamma/(1-\gamma)}}{\left\{1 + [(\gamma - 1)/2]M_\infty^2\right\}^{\gamma/(1-\gamma)}} = \left\{\frac{1 + [(\gamma - 1)/2]M_c^2}{1 + [(\gamma - 1)/2]M_\infty^2}\right\}^{\gamma/(1-\gamma)} \qquad (10)$$

Also note that the freestream and cowl areas can be related by

$$A_c \big/ A^* = (A_c/A_\infty)\big(A_\infty \big/ A^*\big) \qquad (11)$$

where A^* is the cross-sectional flow area where $M = 1$ for isentropic flow. Then with

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M^2\right)\right]^{(\gamma+1)/2(\gamma-1)} \qquad (12)$$

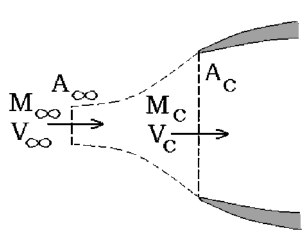

Fig. 1 Inlet nomenclature.

Eqs. (11) and (12) are combined to yield

$$\frac{1}{M_c}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_c^2\right)\right]^{(\gamma+1)/2(\gamma-1)} = \frac{A_c}{A_\infty}\frac{1}{M_\infty}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\right]^{(\gamma+1)/2(\gamma-1)} \tag{13}$$

which relates M_c and M_∞. Equation (13) can be simplified further simplified to

$$\frac{1}{M_c}\left(1+\frac{\gamma-1}{2}{M_c}^2\right)^{(\gamma+1)/2(\gamma-1)} = \frac{A_c}{A_\infty}\frac{1}{M_\infty}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{(\gamma+1)/2(\gamma-1)} \tag{14}$$

Equation (14) is then rearranged and squared to define $(M_c/M_\infty)^2$

$$\left(\frac{M_c}{M_\infty}\right)^2 = \left(\frac{A_\infty}{A_c}\right)^2\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1)/2]M_\infty^2}\right\}^{(\gamma+1)/(\gamma-1)} \tag{15}$$

Equations (10) and (15) are then substituted into Eq. (7) to give

$$Cd_{\text{pre}} = \frac{2}{\gamma M_\infty^2}\left(\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1)/2]M_\infty^2}\right\}^{\gamma/(1-\gamma)} - 1\right) + 2\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1)/2]M_\infty^2}\right\}^{\gamma/(1-\gamma)}\left(\frac{A_\infty}{A_c}\right)^2 \times\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1)/2]M_\infty^2}\right\}^{(\gamma+1)/(\gamma-1)} - 2\left(\frac{A_\infty}{A_c}\right) \tag{16}$$

The second term in Eq. (16) can be simplified such that

$$Cd_{\text{pre}} = \frac{2}{\gamma M_\infty^2}\left(\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1)/2]M_\infty^2}\right\}^{\gamma/(1-\gamma)} - 1\right) + 2\left\{\frac{1+[(\gamma-1)/2]M_c^2}{1+[(\gamma-1/2]M_\infty^2}\right\}^{1/(\gamma-1)}\left(\frac{A_\infty}{A_c}\right)^2 - 2\left(\frac{A_\infty}{A_c}\right) \tag{17}$$

Note that Cd_{pre} is now in terms of γ, M_c, M_∞, and A_∞/A_c. Because of the relationship of M_∞ and M_c as defined by Eq. (14), M_c cannot be eliminated from this definition of Cd_{pre}. However, the limit of the third term on the right-hand side of Eq. (17) as M_∞ goes to zero is $-2(A_\infty/A_c)$. Because M_c goes to zero as M_∞ goes to zero, the second term on the right-hand side of Eq. (17) goes to $2(A_\infty/A_c)^2$. The limit of the first term as M_∞ goes to zero is not easily determined because of M_∞^2 in the denominator. Before applying L'Hopital's rule to this first term, the following intermediate parameters are defined to simplify the results of the application:

$$a \doteq (\gamma-1)/2, \qquad b \doteq (A_c/A_\infty), \qquad c \doteq \gamma/(1-\gamma)$$
$$x \doteq M_\infty, \qquad y \doteq M_c \tag{18}$$

The first term of Eq. (17) can be rewritten using the relations of Eq. (18) as the ratio of two functions of x. Then L'Hopital's rule may be applied twice

$$\lim_{x\Rightarrow 0}\frac{2[(1+ay^2)/(1+ax^2)]^c - 2}{\gamma x^2} = \frac{\lim_{x\Rightarrow 0} f(x)}{\lim_{x\Rightarrow 0} g(x)} = \frac{\lim_{x\Rightarrow 0} f'(x)}{\lim_{x\Rightarrow 0} g'(x)} = \frac{\lim_{x\Rightarrow 0} f''(x)}{\lim_{x\Rightarrow 0} g''(x)} \tag{19}$$

where

$$f(x) = \frac{2[(1+ay^2)/(1+ax^2)]^c - 2}{\gamma x^2}, \qquad g(x) = \gamma x^2 \tag{20}$$

The form of $g(x)$ requires that L'Hopital's rule be used twice so that the limit of $g''(x)$ is not zero, but instead

$$\lim_{x\Rightarrow 0} g''(x) = 2\gamma \tag{21}$$

The remaining task is to determine the limit of $f''(x)$ as M_∞ goes to zero. Careful differentiation of $f(x)$ with respect to x yields

$$f'(x) = 4ac\left(\frac{1+ay^2}{1+ax^2}\right)^c\left[(1+ay^2)^{-1}y\frac{\mathrm{d}y}{\mathrm{d}x} - (1+ax^2)^{-1}x\right] \tag{22}$$

There is no need to determine the limit of $f'(x)$ as x goes to zero because the ratio of the limits of $f'(x)/g'(x)$ is either indeterminate or infinitely large because the denominator approaches zero as a limit. A careful differentiation of Eq. (22) yields $f''(x)$, which consists of nine terms multiplied by a common factor $4ac$:

$$\frac{f''(x)}{4ac} = 2ac\left[-xy(1+ay^2)^{c-1}(1+ax^2)^{-c-1}\frac{\mathrm{d}y}{\mathrm{d}x} + y^2(1+ay^2)^{c-2}(1+ax^2)^{-c}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x^2(1+ay^2)^c \times(1+ax^2)^{-c-2} - xy(1+ay^2)^{c-1}(1+ax^2)^{-c-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \sum_{j=5}^{9} T_j(x,y) \tag{23}$$

Evaluation of the limit requires that the limit of $\mathrm{d}y/\mathrm{d}x$ be determined. Equation (23) shows that if the limit of $\mathrm{d}y/\mathrm{d}x$ is finite, then the limiting value for all four terms will be zero. To obtain the limit of $\mathrm{d}y/\mathrm{d}x$, Eq. (14) will be rewritten using the intermediate variables x, y, and a from Eq. (18) and d defined as

$$d \doteq \frac{\gamma+1}{2(\gamma-1)} \tag{24}$$

so that

$$(1/y)(1+ay^2)^d = (1/b)(1/x)(1+ax^2)^d \tag{25}$$

This equation can de differentiated implicitly with respect to x, and the resulting expression is then solved for $\mathrm{d}y/\mathrm{d}x$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a\,dxy(1/b)(1+ax^2)^{d-1} - (1+ay^2)^d}{2a\,dxy(1+ay^2)^{d-1} - (1/b)(1+ax^2)^d} \tag{26}$$

It can be seen from Eq. (26) that as x and y go to zero that

$$\lim_{x\Rightarrow 0}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{-(1/b)} = b \tag{27}$$

Thus, the four terms in Eq. (23) go to zero as x and y go to zero. The remaining five terms of $f''(x)$ are

$$\sum_{j=5}^{9} T_j(x,y) = (1+ay^2)^{c-1}y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(1+ax^2)^{-c} + (1+ay^2)^{c-1} \times(1+ax^2)^{-c}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2ay^2(1+ay^2)^{c-2}(1+ax^2)^{-c}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - (1+ay^2)^c(1+ax^2)^{-c-1} + 2ax^2(1+ax^2)^{-2-c}(1+ay^2)^c \tag{28}$$

In this case, an expression for d^2y/dx^2 is needed because it appears in Eq. (28). Equation (26) is differentiated a second time with respect to x to determine d^2y/dx^2. Before proceeding with this differentiation, some additional intermediate variables are defined to simplify the results of the differentiation

$$D \doteq 2ad(1/b), \qquad E \doteq 2ad \tag{29}$$

The expression for d^2y/dx^2 is quite lengthy but consists of two terms in the numerator and a squared expression in the denominator:

$$\frac{d^2y}{dx^2} = \frac{FG - HI}{J^2} \tag{30}$$

The first factor F is

$$F = Exy(1 + ay^2)^{d-1} - (1/b)(1 + ax^2)^d \tag{31}$$

This expression goes to $-1/b$ as x and y go to zero. The second factor G is

$$G = Dxy(d-1)(1 + ax^2)^{d-2}2ax + (1 + ax^2)^{d-1}D\left(x\frac{dy}{dx} + y\right)$$
$$- d(1 + ay^2)^{d-1}2ya\frac{dy}{dx} \tag{32}$$

The value of all terms in this expression go to zero as x and y go to zero. Thus, the product FG goes to zero as x goes to zero. The factor H in Eq. (30) is

$$H = Dxy(1 + ax^2)^{d-1} - (1 + ay^2)^d \tag{33}$$

In this case, as x and y go to zero, the value of this expression goes to -1. Factor I in Eq. (30) is

$$I = Exy(d-1)(1 + ay^2)^{d-2}2ay\frac{dy}{dx}$$
$$+ (1 + ay^2)^{d-1}E\left(x\frac{dy}{dx} + y\right) - d\frac{1}{b}(1 + ax^2)^{d-1}2xa \tag{34}$$

The value of this expression becomes zero as x and y go to zero. Thus, the limit of HI is zero. Therefore, the limit of the numerator of Eq. (30) $[FG - HI]$ is zero. The denominator is

$$J^2 = [Exy(1 + ay^2)^{d-1} - (1/b)(1 + ax^2)^d]^2 \tag{35}$$

The limiting value of this expression as x and y go to zero is $(-1/b)^2$. However, because the numerator is zero, therefore,

$$\lim_{x \Rightarrow 0} \frac{d^2y}{dx^2} = \frac{-1/b \cdot 0 - (-1) \cdot 0}{(-1/b)^2} = 0 \tag{36}$$

With the limits for both the first and second derivatives known, the limit of the five terms of $f''(x)$ in Eq. (28) simplifies to

$$\lim_{x \Rightarrow 0} f''(x) = \lim_{x \Rightarrow 0} \sum_{j=5}^{9} T_j(x, y) = 4ac(0 + b^2 + 0 - 1 + 0)$$
$$= 4ac(b^2 - 1) \tag{37}$$

Now Eq. (18) is used to express Eq. (37) in terms of the original variables

$$\lim_{x \Rightarrow 0} f''(x) = 4\left(\frac{\gamma - 1}{2}\right)\left(\frac{\gamma}{1 - \gamma}\right)\left[\left(\frac{A_\infty}{A_c}\right)^2 - 1\right]$$
$$= -2\gamma\left[\left(\frac{A_\infty}{A_c}\right)^2 - 1\right] \tag{38}$$

The limit for $f''(x)$ can now be divided by the limit for $g''(x)$ [Eq. (21)] to determine the first term of the Cd_{pre} expression [Eq. (17)],

$$\frac{\lim_{x \Rightarrow 0} f''(x)}{\lim_{x \Rightarrow 0} g''(x)} = \frac{-2\gamma\left[(A_\infty/A_c)^2 - 1\right]}{2\gamma} = 1 - \left(\frac{A_\infty}{A_c}\right)^2 \tag{39}$$

Final Result

This third term limit is then added to the limiting values for the other two terms in Eq. (17) to give

$$Cd_{\text{pre}(M_\infty = 0)} = 1 - \left(\frac{A_\infty}{A_c}\right)^2 + 2\left(\frac{A_\infty}{A_c}\right)^2 - 2\left(\frac{A_\infty}{A_c}\right)$$
$$= 1 + \left(\frac{A_\infty}{A_c}\right)^2 - 2\left(\frac{A_\infty}{A_c}\right) = \left(1 - \frac{A_\infty}{A_c}\right)^2 \tag{40}$$

Note that this simple result is independent of the value of γ. This result agrees with the data shown in Fig 9.3 of Ref. 1.

Conclusion

A simple expression, independent of γ, for determining the additive drag coefficient at static conditions has been derived from an existing expression which predicts the additive drag coefficient at non-static conditions.

Acknowledgment

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Intercusp Electron Transport in an NSTAR-Derivative Ion Thruster

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Introduction

FOR typical magnetic flux densities present in conventional ion thruster discharge chambers, electron motion is influenced by the presence of the magnetic field. In a well-designed ring-cusp discharge chamber, the magnetic field between the magnetic cusps provides a fairly high impedance path for electron flow to the anode, thereby reducing electron losses to intercusp regions.[1] In this respect, it is not surprising that many of the past studies regarding electron transport to the anode in discharge chambers have concentrated on the magnetic cusps where most of the electron current is collected.[1–5] Recently, however, in an effort to save on launch costs as well as to satisfy requirements for smaller spacecraft missions, low-mass ion thrusters are being developed.[6–10] Ion thruster mass reductions can be achieved by reducing the number of magnet rings in the ring-cusp magnetic circuit and by using lightweight, nonmagnetic materials such as aluminum and titanium for discharge chamber construction. The NSTAR ion thruster, which provided the primary propulsion for the Deep Space 1 mission, is an example of such an engine.[11,12] The NSTAR ion thruster featured only three magnet rings and a discharge chamber constructed of titanium and

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